GROUP PROPERTIES OF THE EQUATIONS OF A SINGLE JET STREAM OF AN IDEAL FLUID

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As is well known, considerable mathematical difficulties are encountered in study of the three-dimensional motions of an ideal fluid with free surfaces, since for this case we do not have a powerful mathematical tool at our disposal such as the theory of complex variables, used in the case of two-dimensional jet streams [1]. Thus, it is of interest to study a particular form of three-dimensional jet streams of an ideal fluid propagating in a thin layer over an arbitrary hard surface [2–4]. Fluid motions of this type are observed when a hydromonitor jet flows over rock, in the operation of Pelton hydroturbines [2], in equipment for determination of dynamic surface tension [5,6], in the theory of the jet flap and gashydrodynamic vane [7–9], in the atomization of fuel by injectors [10], and in various kinds of equipment encountered in chemical engineering [11,12].

In this paper the author investigated the group properties of equations for the axisymmetric jet flow of a thin layer of weightless fluid propagating over an arbitrary surface of rotation, and he constructed some invariant solutions for these equations.

1. Consider a thin-layer jet stream of an ideal incompressible fluid flowing over an impenetrable smooth surface of rotation S. This motion is referred to a curvilinear orthogonal coordinate system connected directly to the surface over which the flow passes. An orthogonal family of curvature lines [2] (parallels and meridians) is chosen for the coordinate lines q_1 = const and q_2 = const on the surface S. The third coordinate q_3 is taken in the direction of the external normal of the surface S. In this system of coordinates the Euler equations and the equation of continuity for steady-state axisymmetric flow have the form

$$\frac{v_1}{H_1}\frac{\partial v_1}{\partial q_1}+\frac{v_3}{H_3}\frac{\partial v_1}{\partial q_3}+\frac{v_1v_3}{H_1H_3}\frac{\partial H_1}{\partial q_3}-\frac{v_3^2}{H_1H_3}\frac{\partial H_3}{\partial q_1}=-\frac{1}{\rho H_1}\frac{\partial p}{\partial q_1},$$
$$\frac{v_3}{H_3}\frac{\partial v_3}{\partial q_3}+\frac{v_1}{H_1}\frac{\partial v_3}{\partial q_1}+\frac{v_1v_3}{H_1H_3}\frac{\partial H_3}{\partial q_1}-\frac{v_1^2}{H_1H_3}\frac{\partial H_1}{\partial q_3}=-\frac{1}{\rho H_3}\frac{\partial p}{\partial q_3},$$
$$\frac{\partial}{\partial q_1}(v_1H_2H_3)+\frac{\partial}{\partial q_3}(v_3H_1H_2)=0. \tag{1.1}$$

Here v_1 and v_3 are velocity components in the direction of the coordinate axes, p and ρ are the pressure and density of the fluid, respectively, and H_1, H_2, and H_3 are the Lamé parameters, specified in this case [13] by the formulas

$$H_1(q_1,q_3)=A_1(q_1)\,[1+q_3/R_1(q_1)],$$
$$H_2(q_1,q_3)=A_2(q_1)\,[1+q_3/R_2(q_1)],$$
$$H_3(q_1,q_3)=1. \tag{1.2}$$

In formulas (1.2) $R_1(q_1)$ and $R_2(q_1)$ are the principal radii of curvature, while $A_1(q_1)$ and $A_2(q_1)$ are the coefficients of the first principal quadratic form $dS^2=A_1^2(q_1)dq_1^2+A_2^2(q_1)dq_2^2$ for a surface of rotation. Let the surface S be formed by rotation of the two-dimensional curve $z=z(s)$, $y=y(s)>0$ around the z-axis of a Cartesian xyz-coordinate system (s is the arc length of the curve). In the case the first principal quadratic form for the surface of rotation S is [14]

$$dS^2=ds^2+y^2(s)dq_2^2,\ A_1=1,\ A_2=y(s),\ s=q_1. \tag{1.3}$$

If we assume that the jet thickness h(s), measured along the normal to the surface S until the intersection with the free surface of the jet, is considerably less than $R=\min\{R_1(s),R_2(s)\}$ on the surface, then for axisymmetric flow, from Eqs. (1.1) and similar to the two-dimensional case [4], we have

$$u\frac{\partial u}{\partial s}+v\frac{\partial u}{\partial r}=-\frac{\partial w}{\partial s},\quad u^2\varkappa(s)=\frac{\partial w}{\partial r},$$

$$\varkappa(s)=\frac{1}{R_1(s)},\qquad \frac{\partial u}{\partial s}+\frac{\partial v}{\partial r}+u\frac{y'(s)}{y}=0, \tag{1.4}$$

$$v_1=u,\qquad v_3=v,\qquad p/\rho=w,\qquad q_3=r. \tag{1.5}$$

Equations (1.4) are the limiting form of system (1.1) as $h/R\to0$.

2. Let us investigate the group properties of the system of quasilinear differential equations (1.4).

With the method of paper [15] we calculate the principal group G of system (1.4). Group G is completely determined by the Lie algebra of its infinitesimal operators

$$X=\xi_s\frac{\partial}{\partial s}+\xi_r\frac{\partial}{\partial r}+\xi_u\frac{\partial}{\partial u}+\xi_v\frac{\partial}{\partial v}+\xi_w\frac{\partial}{\partial w}. \tag{2.1}$$

Here ξ_s, ξ_r, ξ_u, ξ_v, and ξ_w are functions of s, u, v, w, and r.

In the general case the solution of the defining equations for the Lie algebra of system (1.4) depends on three arbitrary constants, and so group G is the product of a three-parameter group, generated by the independent operators

$$X_1=\frac{\partial}{\partial r},\quad X_2=\frac{\partial}{\partial w},\quad X_3=u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+2w\frac{\partial}{\partial w}. \tag{2.2}$$

A greater number of operators is possible only for the case in which the surface of rotation over which the flow passes is such that the following relation is satisfied:

$$y'(s)\ \varkappa(s)/y(s)\varkappa'(s)=b=\text{const}. \tag{2.3}$$

Besides operators (2.2), system (1.4) permits one additional operator in this case:

$$X_4=-\frac{\varkappa}{\varkappa'}\frac{\partial}{\partial s}+r\frac{\partial}{\partial r}+\left(1+\frac{d}{ds}\frac{\varkappa}{\varkappa'}\right)v\frac{\partial}{\partial v}, \tag{2.4}$$

and G is a four-parameter group.

Any transformation of group G can be obtained by a certain superposition of the transformations corresponding to the operators X_1, X_2, X_3, and X_4. It can easily be verified that the set of operators with base (2.2) and (2.4) forms a Lie algebra. Calculations show that a commutator of any two operators from (2.2) and (2.4) is a linear combination of the same operators with constant coefficients.

Let us investigate invariant solutions of system (1.4), constructed on single-parameter subgroups of the basic group G. It can easily be shown that to determine all those essentially different invariant solutions of system (1.4) constructed on single-parameter subgroups of the principal group G, it suffices to construct solutions on the following subgroups, which form an optimum system of single-parameter subgroups of the principal group G for system (1.4) in the case of an arbitrary surface of rotation. These subgroups are represented by

$$X_1,\qquad X_1+X_2,\qquad X_2,\quad \alpha X_1+X_3. \tag{2.5}$$

Here α is an arbitrary parameter.

For the surface of rotation satisfying (2.3), the optimum system has the form

$$X_1,\quad X_1+X_2,\quad X_2,\quad \alpha X_1+X_3,$$
$$X_2+X_4,\quad \beta X_3+X_4. \tag{2.6}$$

Here β is an arbitrary parameter.

3. In this section we discuss some of the most interesting invariant solutions constructed on the subgroups of (2.5) and (2.6).

(a) Operator αX_1+X_3 in (2.5). A full set of functionally independent invariants [15] of this subgroup is

$$J_1=ue^{-r/\alpha},\qquad J_2=ve^{-r/\alpha},$$

$$J_3 = we^{-2r/\alpha}, \qquad J_4 = s \quad (\alpha \neq 0). \tag{3.1}$$

Solving (3.1) for u, v, and w, we have

$$u = J_1(s)\, e^{r/\alpha}, \quad v = J_2(s)\, e^{r/\alpha}, \quad w = J_3(s)\, e^{2r/\alpha}. \tag{3.2}$$

Substituting (3.2) into the equations of system (1.4) we obtain for the functions $J_1(s)$, $J_2(s)$, and $J_3(s)$ the following system of ordinary differential equations:

$$\alpha J_1 J_1' + J_2 J_1 + \alpha J_3' = 0,\ \alpha J_1^2 \varkappa = \\ = 2J_3,\ \alpha J_1' + J_2 + \alpha J_1\, y'/y = 0. \tag{3.3}$$

System (3.3) can be solved. We write the solution of the principal equations (1.4) in the form

$$u = \left(\frac{2c_1}{\alpha\varkappa}\right)^{1/2} \exp\left\{\frac{r}{\alpha} - \int \frac{y'ds}{\alpha y\varkappa}\right\},$$

$$v = -\left(\frac{2c_1}{\alpha\varkappa}\right)^{1/2} \exp\left\{\frac{r}{\alpha} - \int \frac{y'ds}{\alpha y\varkappa}\right\}\left(\frac{\alpha y'}{y} - \frac{\alpha\varkappa'}{2\varkappa} + \frac{y'\sqrt{\alpha}}{y\varkappa}\right),$$

$$w = c_1 \exp\left\{\frac{2r}{\alpha} - \int \frac{2y'ds}{\alpha y\varkappa}\right\} \quad (c_1 = \text{const}). \tag{3.4}$$

We require that solution (3.4) satisfy the condition

$$v = 0 \quad \text{for} \quad r = 0, \tag{3.5}$$

i.e., the normal component of velocity v should vanish at the surface over which the flow passes. For $c_1 \neq 0$ we find from (3.4) that the solution constructed has a meaning in the study of jet motions only over a surface defined by the equation

$$\frac{\alpha\varkappa'}{2\varkappa} = \frac{\alpha y'}{y} + \frac{y'\sqrt{\alpha}}{y\varkappa}, \tag{3.6}$$

where the curvature $\varkappa(s)$ can be expressed by $y(s)$ [14]. In this case $v \equiv 0$ in solution (3.4).

The condition

$$p = p_0 \ \text{for} \ r = h(s) \tag{3.7}$$

at the free surface of the jet determines h(s), the jet thickness. When (3.6) is used, the last relation of (3.4) gives us

$$h(s) = \frac{\alpha}{2} \ln \frac{p_0}{\rho c_1}\left(\frac{\varkappa\sqrt{\alpha}}{\varkappa\sqrt{\alpha}+1}\right)^{\frac{1}{\alpha\sqrt{\alpha}}}. \tag{3.8}$$

Integrating (3.6) and remembering that

$$dy^2 + dz^2 = ds^2,$$

we obtain the equation for the generatrix of the surface of rotation in quadrature form

$$z = \int \frac{D_1 y^{2\alpha+1} - y + D_2}{[1 - (D_1 y^{2\alpha+1} - y + D_2)^2]^{1/2}}\, dy + D_3, \tag{3.9}$$

where D_1, D_2, and D_3 are constants which are determined by specifying the coordinates, angle of inclination, and curvature of the generatrix at the origin.

(b) Operator $\beta X_3 + X_4$ in (2.6). In this case a complete set of functionally independent invariants is

$$J_1 = r\varkappa(s), \quad J_2 = ur^{-\beta}, \quad J_3 = wr^{-2\beta}, J_4 = v\varkappa^{\beta+2}/\varkappa'. \tag{3.10}$$

Solving (3.10) for u, v, and w, we obtain

$$u = J_2(\xi)\, r^{\beta}, \quad v = J_4(\xi)\varkappa'(s)/\varkappa^{\beta+2},$$

$$w = J_3(\xi)\, r^{2\beta}, \quad \xi = J_1 = r\varkappa(s) \tag{3.11}$$

To determine the functions $J_2(\xi)$, $J_3(\xi)$, and $J_4(\xi)$ from Eqs. (1.4) and relation (2.3) we have the system of ordinary differential equations:

$$J_2 J_2'\xi + J_4 J_2'\xi^{-\beta} + J_3'\xi = 0,$$

$$J_2^2 = J_3' + 2\beta J_3 \xi^{-1}, \qquad J_2'\xi + J_4'\xi^{-\beta} + J_2 b = 0 \tag{3.12}$$

For the case $\beta = 0$ Eqs. (3.12) are integrated in finite form and we obtain

$$u = c_2 \exp\left\{-\int \frac{2\xi d\xi}{\xi^2 - 2(1-b)\xi + 2c_3}\right\},$$

$$v = \frac{c_2\varkappa'}{2\varkappa^2}(\xi^2 - 2b\xi + 2c_3)\exp\left\{-\int \frac{2\xi d\xi}{\xi^2 - 2(1-b)\xi + 2c_3}\right\},$$

$$w = c_2^2 \int \exp\left\{-\int \frac{4\xi d\xi}{\xi^2 - 2(1-b)\xi + 2c_3}\right\} d\xi + c_4. \tag{3.13}$$

Here c_2, c_3, and c_4 are integration constants. It follows from condition (3.5) that $c_3 = 0$.

Thus a particular solution of system (1.4) describing the jet motion of a thin layer of fluid over a surface of rotation, takes the form

$$u = \frac{c_2}{[\xi + 2(1-b)]^2}, \quad v = \frac{c^2\varkappa'}{2\varkappa}\frac{r(\xi - 2b)}{[\xi + 2(1-b)]^2},$$

$$w = c_4 - \frac{c_2^2}{3[\xi + 2(1-b)]^3}. \tag{3.14}$$

This solution is valid for a surface of rotation which satisfies relation (2.3), from which we find the equation for the generatrix of the surface of rotation in the form:

$$z = \int \frac{F_1 y^m + F_2}{[1 - (F_1 y^m + F_2)^2]^{1/2}}\, dy + F_3, \tag{3.15}$$

where F_1, F_2, and F_3 are integration constants and $m = (b+1)b^{-1}$.

As before, the jet thickness is determined by (3.7), the condition at the free surface.

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